

Non Degenerate Ultrametric Diffusion

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Abstract. General non-degenerate p -adic operators of ultrametric diffusion are introduced. Bases of eigenvectors for the introduced operators are constructed and the corresponding eigenvalues are computed. Properties of the corresponding dynamics (i.e. of the ultrametric diffusion) are investigated.

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1. Introduction

Ultrametric diffusion models were investigated in relation with models of complex systems, see for instance [1, 2]. Mathematical theory of ultrametric diffusion was investigated in [3–5]. In the recent works [6–8], p -adic models of ultrametric diffusion has been discussed in connection with description of protein dynamics and characteristic types of relaxation in complex systems. Ultrametric diffusion models are naturally related to the basin-to-basin kinetics approach proposed rather long time ago. Basin-to-basin kinetics approach was widely used in computer study of the dynamics constrained by rough multidimensional energy landscapes [9–14].

The basin-to-basin kinetics approach can be outlined as follows. Let us consider a system, which is described by a particle performing random walk in a rough energy landscape. The system is supposed to be arriving to the nearest quasi-equilibrium state (the nearest local minimum on the energy landscape) from any initial state in the time, which is much smaller compared to the lifetime of this quasi-equilibrium state. Therefore, we will reduce our consideration to the set of local minima of the energy landscape. Further, we assume that the set of local minima can be represented as a union of hierarchically nested subsets. These subsets we will call the basins of minima. Each of the basins is a union of the non-overlapping basins of the smaller size (subbasins), each of these smaller basins is a union of the still smaller ones etc. Moreover, we assume that the larger basins are separated by the higher activation barriers, and the smaller subbasins are separated by the lower barriers, i.e. if the basin A is a subbasin of the basin B , then the activation barriers between the maximal subbasins of A is smaller then the activation barriers between the maximal subbasins of B . The basin hierarchy corresponds to the hierarchy of the configuration rearrangements, and the hierarchy of the activation barriers corresponds to the hierarchy of characteristic times of these rearrangements.

As a result, the multidimensional energy surface can be represented by a tree — a "skeleton" of a hierarchical landscape [10, 11, 13, 15] (Fig. 1). This tree reflects the hierarchy of nestings and it is directed (i.e. it is a tree with a partial order). The vertices of the tree correspond to the basins, the partial order describes the ordering of the basins (i.e. the vertex A is larger than B if the corresponding basin A contains the basin B). The local minima of the landscapes correspond to minimal vertices with respect to the introduced ordering. In general case for finite or infinite tree the set of local minima will be described by the absolute of the tree. Absolute of a tree is the set of equivalence classes of decreasing paths in the tree. The equivalence class contains all the decreasing paths, where each two paths coincide starting from some vertex.

Absolute of any tree has a natural structure of ultrametric space, see [16]. Description of a rough energy landscape in terms of hierarchically nested basins is equivalent to the introduction of ultrametric space of states. Thus, the basin-to-basin kinetics can be understood as a diffusion (more definitely, as a jump process) in the ultrametric space [6–8]. The transition probability T_{xy} between the states y and x is

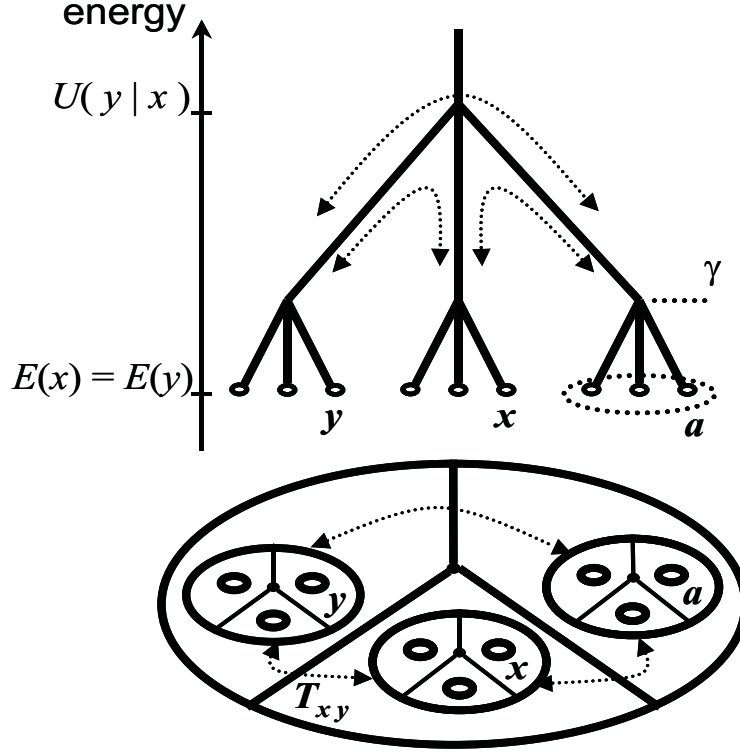


Figure 1. Hierarchy of basins and activation barriers in the basin-to-basin kinetic approach. For explanation of the notations see the text.

determined by the position of the vertex $U(y|x)$ in which the tree branches into the paths going to the points x and y , and by the energies $E(x)$ and $E(y)$ of the states x and y (Fig. 1). In particular, if all the local minima have the same energy, then T_{xy} is determined only by the vertex $U(y|x)$, i.e. the transition probabilities has the property of permutation symmetry $T_{xy} = T_{yx}$.

The well known example of an ultrametric space is the field of p -adic numbers \mathbb{Q}_p . In the works [6–8], p -adic description of ultrametric diffusion is introduced in the following way. The system states are parameterized by the p -adic coordinate x , a basin of states corresponds to the p -adic disk $B_\gamma(a)$ (Fig. 1). The p -adic disk $B_\gamma(a)$ is a set of all p -adic numbers $\{x : |x - a|_p \leq p^{-\gamma}\}$, for which the p -adic distance from the disk center a ($a \in \mathbb{Q}_p$) is less or equal than the radius $p^{-\gamma}$, where γ is an integer ($\gamma \in \mathbb{Z}$). The parameters γ and a distinguish the p -adic disks $B_\gamma(a)$.

To describe the system evolution we introduce the probability distribution function $f(x, t)$ which depends on the p -adic coordinate x and the real time t : the integral

$$\int_B f(x, t) d\mu(x)$$

($d\mu(x)$ is the Haar measure on \mathbb{Q}_p) is the probability to find the system in the set B at time t .

The evolution of the function $f(x, t)$ is described by the equation

$$\frac{\partial f(x, t)}{\partial t} = - \int_{\mathbb{Q}_p} (T_{yx}f(x, t) - T_{xy}f(y, t)) d\mu(y) \quad (1)$$

This is the master equation for ultrametric diffusion, and the linear integral operator at the RHS of (1)

$$Tf(x) = \int_{\mathbb{Q}_p} (T_{yx}f(x) - T_{xy}f(y)) d\mu(y) \quad (2)$$

is called the operator of ultrametric diffusion.

The non-negative kernel T_{xy} is equal to the rate of transition from the state y to the state x ($T_{xy} : \mathbb{Q}_p \times \mathbb{Q}_p \mapsto \mathbb{R}_+$). We consider the kernels T_{xy} , which are locally constant outside any vicinity of $x = y$. The complex valued function $g(x)$, defined on \mathbb{Q}_p , is called locally constant, if

$$\forall x \in \mathbb{Q}_p \quad \exists \gamma \in \mathbb{Z} \quad \forall z \in \mathbb{Q}_p : |z|_p \leq p^\gamma \Rightarrow g(x+z) = g(x)$$

The sets, on which the function is constant, are called the sets of the local constancy of the function.

In the simplest case the operators of ultrametric diffusion can be introduced by the kernels T_{xy}^0 satisfying the following conditions:

(i)

$$\begin{aligned} \text{for fixed } y \quad \forall x, z \in \mathbb{Q}_p : |z|_p \leq |x - y|_p &\Rightarrow T_{(x+z)y}^0 = T_{xy}^0 \\ \text{for fixed } x \quad \forall y, z \in \mathbb{Q}_p : |z|_p \leq |x - y|_p &\Rightarrow T_{x(y+z)}^0 = T_{xy}^0 \end{aligned} \quad (3)$$

(ii)

$$T_{xy}^0 = T_{yx}^0$$

i.e. the kernel T_{xy}^0 is symmetric (and the corresponding operator is Hermitian, since T_{xy}^0 is a real-valued function).

(iii)

$$\forall a \in \mathbb{Q}_p \quad T_{xy}^0 = T_{x-a, y-a}^0$$

i.e. the kernel T_{xy}^0 has the property of translation invariance.

In the following we will denote by T^0 the operator of ultrametric diffusion, and by T_{xy}^0 the corresponding kernel. The general form of the kernels T_{xy}^0 is given by the series

$$T_{xy}^0 = \sum_{\gamma=-\infty}^{\infty} T^{(\gamma)} \delta_{p^\gamma, |x-y|_p} \quad (4)$$

where δ is the Kronecker delta. The kernels T^0 can be equivalently described by the functions dependent only on p -adic norm of the difference of x and y : $T_{x,y}^0 = \rho(|x-y|_p)$. The coefficients $T^{(\gamma)}$ of the series (4) and the function $\rho(|x-y|_p)$ are connected by the relation $T^{(\gamma)} = \rho(p^\gamma)$.

The properties (i) - (iii) allow us to use the p -adic Fourier transformation to compute the eigenvalues of the operator. If the series $\sum_{\gamma=0}^{\infty} p^{\gamma} T^{(\gamma)}$ converges, the eigenvalues of the operator T^0 are determined by the expression

$$\lambda_{\gamma}^0 = p^{\gamma} T^{(\gamma)} + (1 - p^{-1}) \sum_{\gamma'=\gamma+1}^{\infty} p^{\gamma'} T^{(\gamma')}$$

Every eigenvalue λ_{γ}^0 is infinitely degenerate.

The operators of the form T^0 were discussed in the context of p -adic mathematical physics. When the kernel has the form with

$$T_{xy} = \frac{p^{\alpha} - 1}{1 - p^{-1-\alpha}} |x - y|_p^{-(1+\alpha)}, \quad \alpha > 0$$

the operator T^0 is the Vladimirov operator of p -adic fractional derivation [3]. Its eigenvalues are given by $\lambda_{\gamma} = p^{(1-\gamma)\alpha}$, $\gamma \in \mathbb{Z}$. Different examples of the operators T^0 have been recently investigated in [8].

In the context of basin-to-basin kinetics, the operator symmetry (hermiticity) property means that all the local minima of the energy landscape have equal energy. The translation invariance of the kernel means that the transition between x and y depends only on the ultrametric distance between x and y .

The translationally invariant operators T^0 are related to the Parisi matrices (see [6, 17]) which were used in the replica approach to spin glasses [18]. However, the energy landscapes of many other disordered systems (for instance, the energy landscapes of clusters, macromolecular structures and biopolymers, see for example [11]) do not have such special properties. Therefore, the ultrametric diffusion operators, different from T^0 , are of great importance.

Generally, the ultrametric diffusion operator T can be defined by (2) where the kernel satisfies some weaker conditions than conditions (i)–(iii). In the present paper we consider translationally *noninvariant* operators of ultrametric diffusion satisfying the hermiticity property. We will examine two types of such operators, T^I and T^{II} .

A family of operators of p -adic diffusion T^I has been recently investigated in the paper [19]. The local constancy conditions for these operators are given by (3), as well as for T^0 , but T^I differs from T^0 by violating the condition (iii).

The kernel T_{xy}^I is described by the expression

$$T_{xy}^I = \sum_{\gamma=-\infty}^{\infty} \sum_{n \in \mathbb{Q}_p / \mathbb{Z}_p} T^{(\gamma n)} \delta_{1, |p^{\gamma} x - p^{\gamma} y|_p} \Omega(p^{\gamma} x - n) \quad (5)$$

where the factorgroup $\mathbb{Q}_p / \mathbb{Z}_p$ is identified with a set of the fractions $\sum_{\gamma=1}^k n_{\gamma} p^{-\gamma}$, $n_{\gamma} = 0, \dots, p-1$, k is any natural number, and the coefficients $T^{(\gamma n)} \geq 0$. The function $\Omega(|x|_p)$ is an indicator of the p -adic disk

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \leq 1 \\ 0, & |x|_p > 1 \end{cases}$$

It was shown that if the series $\sum_{\gamma'=\gamma}^{\infty} p^{\gamma'} T^{(\gamma'n)}$ converges, the eigenvalues of the corresponding operator are given by

$$\lambda_{\gamma n}^I = p^{\gamma} T^{(\gamma n)} + (1 - p^{-1}) \sum_{\gamma'=\gamma+1}^{\infty} p^{\gamma'} \sum_{n' \in \mathbb{Q}_p / \mathbb{Z}_p} T^{(\gamma'n')} \delta_{n', np^{\gamma'}} \quad (6)$$

and, in general, are $p - 1$ times degenerate. The eigenvalue $\lambda_{\gamma n}^I$ corresponds to the eigenvectors

$$\varphi_{\gamma n j}(x) = p^{-\gamma/2} \chi_p(p^{\gamma-1} j x) \Omega(|p^{\gamma} x - n|_p) \quad (7)$$

where $j = 1, \dots, p-1$. Here the function $\chi_p(x) = \exp(2\pi i \{x\}_p)$ is an additive character on the field \mathbb{Q}_p , the symbol $\{x\}_p$ denotes a fractional part of the p -adic number. Recall that if the canonical decomposition of the p -adic number x is given by

$$x = p^{\gamma} \sum_{\mu=0}^{\infty} x_{\mu} p^{\mu}, \quad x_{\mu} = 0, \dots, p-1, \quad x_0 \neq 0 \quad (8)$$

then the fractional part of the number x is determined by the following expression

$$\{x\}_p = \begin{cases} 0, & \gamma \geq 0 \\ p^{\gamma} (x_0 + x_1 + \dots + x_{|\gamma|-1} p^{|\gamma|-1}), & \gamma < 0 \end{cases}$$

As shown in [20], the set of vectors (7) forms an orthonormal basis in $L^2(\mathbb{Q}_p)$, which was called there the basis of p -adic wavelets.

In the present paper, in section 2, we propose a new general expression for the kernel T^I and show the equivalence of this expression with (5).

In section 3 we introduce the new type of transitionally noninvariant kernel, T^{II} (more general compared to T^I), satisfying the weaker (compared with (3)) condition of local constancy:

$$\begin{aligned} \text{for fixed } y \quad \forall x, z \in \mathbb{Q}_p : |z|_p \leq p^{-1} |x - y|_p &\Rightarrow T_{(x+z)y}^{II} = T_{xy}^{II} \\ \text{for fixed } x \quad \forall y, z \in \mathbb{Q}_p : |z|_p \leq p^{-1} |x - y|_p &\Rightarrow T_{x(y+z)}^{II} = T_{xy}^{II} \end{aligned} \quad (9)$$

We introduce the kernel T^{II} both in the functional form and in the form of a series, and we show the equivalence of these definitions. In this section, we find the eigenfunctions of the introduced operator, which form the basis in $L^2(\mathbb{Q}_p)$, compute the corresponding eigenvalues, and show that the eigenvalues, in general, are non degenerate.

In section 4, we investigate the properties of ultrametric diffusion, generated by the operators T^I and T^{II} . We consider the relaxation of the initially localized state. We show that the inhomogeneities of the landscape described by the translationally noninvariant kernels T^I and T^{II} are not essential for long-time relaxation, i.e. the asymptotics for the relaxation, correspondent to kernels $T_{xy}^0, T_{xy}^I, T_{xy}^{II}$ will be the same for the cases when the corresponding kernels are naturally related in the way described in Section 4..

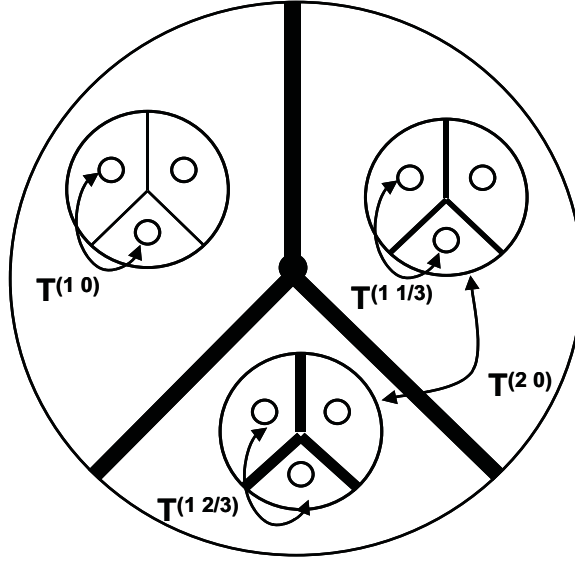


Figure 2. Scheme of the transitions corresponding to the operator T^I with $p = 3$. The lines of different thickness correspond to the different transition probabilities $T^{(\gamma n)}$ between the p -adic disks, which are marked by the circles.

2. Translationally noninvariant operators of ultrametric diffusion T^I

In the present section we propose a new expression for the kernel T^I and show that this new expression is equivalent to the already known one. Expression (5) implies, that if $x \in B_\gamma(p^{-\gamma n})$ and y satisfies the condition $|x - y|_p = p^\gamma$, then the kernel T^I_{xy} is equal to $T^{(\gamma n)}$. In this case, we can illustrate the basin-to-basin transitions with the help of the scheme on Fig. 2. The given transition scheme allows to construct the expression of the operator T^I . To construct a new expression for the kernel T^I_{xy} we use the following approach. The integration kernel T^I_{xy} of the operator should depend on the two arguments: (i) on the size of the minimal disk containing x and y , which is equal to the distance $|x - y|_p$; (ii) on the argument distinguishing the disk among the other disks of the same size. For the last purpose we fix the disk center. Since any point belonging to an ultrametric disk is its center, we will take as the center of the minimal disk, containing x and y the following

$$\frac{\{x|x - y|_p\}_p}{|x - y|_p} = \begin{cases} \sum_{\mu=\log_p|x-y|_p+1}^{\log_p|x|_p} p^{-\mu} x_\mu, & |x|_p > |x - y|_p \\ 0, & |x|_p \leq |x - y|_p \end{cases} \quad (10)$$

where x_μ are the coefficients of the canonical decomposition (8) of the p -adic number x . Therefore, the kernel of the operator T^I can be represented by the function

$$\rho\left(|x - y|_p, \{x|x - y|_p\}_p\right) \quad (11)$$

The function (11) obviously do not have the translational invariance property. For this function the following proposition is satisfied.

Proposition 1 *The function (11) is symmetric with respect to the $x \mapsto y$, $y \mapsto x$ permutation. The function (11) satisfies the conditions (3) and (5). Moreover, any*

function satisfying the condition (5) can be represented in the form (11). Therefore the representations of the kernel T^I in the forms (11) and (5) are equivalent, and the equivalence is given by the relation

$$T^{(\gamma n)} = \rho(p^\gamma, n), \quad \gamma \in \mathbb{Z}, \quad n \in \mathbb{Q}_p/\mathbb{Z}_p \quad (12)$$

Proof Let us prove the permutation symmetry: $\{y|x-y|_p\}_p = \{x|y-x|_p\}_p$. It follows that

$$\{x|x-y|_p\}_p - \{y|x-y|_p\}_p = \{(x-y)|x-y|_p\}_p = 0$$

For proving the rest of the proposition, we will show at first that any function of the form (11) can be represented in the form (5). Actually, for any $x \in B_\gamma(p^{-\gamma}n)$ ($n \in \mathbb{Q}_p/\mathbb{Z}_p$) and any $y \in \mathbb{Q}_p$ satisfying the condition $|x-y|_p = p^\gamma$, we have $\{x|x-y|_p\}_p = \{n\}_p = n$. Hence,

$$\rho\left(|x-y|_p, \{x|x-y|_p\}_p\right) = \sum_{\gamma=-\infty}^{\infty} \sum_{n \in \mathbb{Q}_p/\mathbb{Z}_p} \rho(p^\gamma, n) \delta_{1, |p^\gamma x - p^\gamma y|_p} \Omega(p^\gamma x - n)$$

This means that the function (11) can be represented in the form of the series (5) with the coefficients $T^{(\gamma n)} = \rho(p^\gamma, n)$.

On the other hand, since no special restrictions are imposed on the function (11), then for all $\gamma \in \mathbb{Z}$ and $n \in \mathbb{Q}_p/\mathbb{Z}_p$ we can put $\rho(p^\gamma, n) = T^{(\gamma n)}$. Therefore the representations of the kernel T^I in the forms (11) and (5) are equivalent, and the equivalence is given by (12).

From the equivalence, it follows that the functions (11) and (5) have the same properties. In particular, the function (11) satisfies the condition (3). \square

3. Translationally noninvariant operators of ultrametric diffusion T^{II} and the basis of generalized p -adic wavelets

Define the family of the operators of ultrametric diffusion T^{II} , with locally constant kernels of the most general form

$$T_{xy}^{II} = \sum_{\gamma=-\infty}^{\infty} \sum_{n \in \mathbb{Q}_p/\mathbb{Z}_p} \sum_{\substack{j,k=0 \\ k \neq j}}^{p-1} T^{(\gamma n j k)} \Omega(p|p^\gamma x - n - j|_p) \Omega(p|p^\gamma y - n - k|_p) \quad (13)$$

where $T^{(\gamma n j k)} = T^{(\gamma n k j)} \geq 0$.

Theorem 2 *The function of the form (13) is symmetric with respect to permutation of the arguments, positive, and satisfies the condition (9).*

Moreover, an arbitrary positive symmetric function satisfying (9) can be represented in the form (13).

Proof Positivity of T_{xy}^{II} is obvious. Permutation symmetry is obvious.

Prove that T_{xy}^{II} given by (13) satisfies (9). This property is easy to check for any product of two indicator functions in (13). By linearity, this proves that T_{xy}^{II} satisfy (9).

Vice versa, it is easy to see that the kernel (13) for x, y lying in the disks with the center in $p^{-\gamma}(n+j)$ and $p^{-\gamma}(n+k)$ correspondingly and the radius $p^{\gamma-1}$, takes the value $T^{(\gamma n j k)}$.

Since all the space $x, y \in \mathbb{Q}_p \times \mathbb{Q}_p$ is the disjoint union of such subsets, therefore, taking an arbitrary positive and symmetric with respect to j, k coefficients $T^{(\gamma n j k)}$, we are able to construct an arbitrary symmetric positive kernel satisfying (9). This finishes the proof of the theorem. \square

The kernel T^{II} can be equivalently described in the functional form.

Proposition 3 *The function*

$$\rho \left(|x - y|_p, \{x|x - y|_p\}_p, \{xp^{-1}|x - y|_p\}_p, \{yp^{-1}|x - y|_p\}_p \right) \quad (14)$$

satisfies the condition (9). Moreover, any function satisfying the condition (9) can be represented in the form (14).

Under condition of symmetry of the function (14) with respect to permutation of x and y , the kernel T^{II} can be equivalently represented in the forms (14) and (13), and the equivalence is given by the relation

$$T^{(\gamma n j k)} = \rho \left(p^\gamma, n, p^{-1}(n+j), p^{-1}(n+k) \right) \quad (15)$$

Proof Let us show first that any function of the form (14) can be represented in the form (13). Actually, any x and y lying in the disks of the radius $p^{\gamma-1}$ with the centers in $p^{-\gamma}n + p^{-\gamma}j$ and $p^{-\gamma}n + p^{-\gamma}k$ ($n \in \mathbb{Q}_p/\mathbb{Z}_p$, $j, k = 0, \dots, p-1$ and $j \neq k$) can be represented in the form

$$x = p^{-\gamma}(n + j + pz_x), \quad |z_x|_p \leq 1$$

$$y = p^{-\gamma}(n + k + pz_y), \quad |z_y|_p \leq 1$$

Whence it follows that at $j \neq k$, $|x - y|_p = p^\gamma$, $\{x|x - y|_p\}_p = n$ and

$$\{xp^{-1}|x - y|_p\}_p = \{p^{-1}(n + j) + z_x\}_p = p^{-1}(n + j)$$

$$\{yp^{-1}|x - y|_p\}_p = \{p^{-1}(n + k) + z_y\}_p = p^{-1}(n + k)$$

Hence,

$$\begin{aligned} \rho \left(|x - y|_p, \{x|x - y|_p\}_p, \{xp^{-1}|x - y|_p\}_p, \{yp^{-1}|x - y|_p\}_p \right) = \\ \sum_{\gamma=-\infty}^{\infty} \sum_{n \in \mathbb{Q}_p/\mathbb{Z}_p} \rho \left(p^\gamma, n, p^{-1}(n+j), p^{-1}(n+k) \right) \Omega(p|p^\gamma x - n - j|_p) \Omega(p|p^\gamma y - n - k|_p) \end{aligned}$$

Thus, the function (14) can be represented in the form of the series (13), whose coefficients are determined by (15).

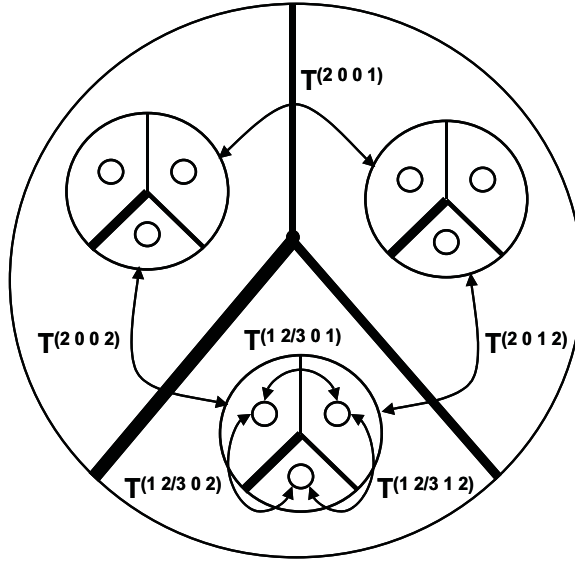


Figure 3. Scheme of the transitions corresponding to the operator T^{II} with $p = 3$.

On the other hand, since no restrictions (except for permutation symmetry) are imposed on the function of the type (14), then for all $\gamma \in \mathbb{Z}$ and $n \in \mathbb{Q}_p/\mathbb{Z}_p$, $j, k = 0, \dots, p-1$ ($j \neq k$) we can put $\rho(p^\gamma, n, p^{-1}(n+j), p^{-1}(n+k))$ to be equal to $T^{(\gamma n j k)}$. Therefore the representations of the kernel in the forms (14) and (13) are equivalent.

Moreover, for the function (14) and the series (13) respectively, the condition of symmetry with respect to permutation of x and y in (14) and of the symmetry of the coefficients $T^{(\gamma n j k)}$ with respect to permutation of j and k , are equivalent.

From this equivalence, it follows that the function (14) satisfies the condition (9), and any function satisfying the condition (9) can be represented in the form (14). \square

Note that, unlike for the operator T^{I} , the kernel of the operator T^{II} formally depends on a couple of additional functions: $\{xp^{-1}|x-y|_p\}_p$ and $\{yp^{-1}|x-y|_p\}_p$. We will explain the meaning of these functions. Consider the transition between the states x and y (see fig. 3). The probability for $x \mapsto y$ transition depends on a relative position of the basins between which the transition is carried out. Let x and y belong to the basin described by the p -adic disk $B_\gamma(a)$ of the radius $p^\gamma = |x-y|_p$ with the center in $a = \{xp^\gamma\}_p p^{-\gamma}$. Then the radii of the disks $B_{\gamma-1}(b)$ and $B_{\gamma-1}(c)$, between which the transition is carried out, are equal to $p^{-1}|x-y|_p$, and the disk centers are determined by the functions $b = \{xp^{\gamma-1}\}_p p^{-\gamma+1}$, $c = \{yp^{\gamma-1}\}_p p^{-\gamma+1}$.

Proposition 4 If $T^{(\gamma n j k)}$ in (13) doesn't depend on j and k , then the operator T^{II} reduces to T^{I} .

Proof Consider the kernel T_{xy}^{II} under the condition that $T^{(\gamma n j k)}$ is independent on j and k :

$$T_{xy}^{\text{II}} = \sum_{\gamma, n} T^{(\gamma n)} \sum_{j=0}^{p-1} \Omega(p|p^\gamma x - n - j|_p) \sum_{\substack{k=0 \\ k \neq j}}^{p-1} \Omega(p|p^\gamma y - n - k|_p) \quad (16)$$

Using the formula

$$\begin{aligned} \Omega(|p^\gamma x - n|_p) &= \Omega(p^{-\gamma}|x - p^{-\gamma}n|_p) \\ &= \sum_{j=0}^{p-1} \Omega(p^{-\gamma+1}|x - p^{-\gamma}n - p^{-\gamma}j|_p) = \sum_{j=0}^{p-1} \Omega(p|p^\gamma x - n - j|_p) \end{aligned} \quad (17)$$

as well as the property of indicators of the disks

$$\Omega(|x - a|_p) \Omega(|y - a|_p) = \Omega(|x - a|_p) \Omega(|x - y|_p)$$

for the sum on j and k in (16) we have

$$\begin{aligned} &\sum_{j=0}^{p-1} \Omega(p|p^\gamma x - n - j|_p) \sum_{\substack{k=0 \\ k \neq j}}^{p-1} \Omega(p|p^\gamma y - n - k|_p) \\ &= \sum_{j=0}^{p-1} \Omega(p|p^\gamma x - n - j|_p) (\Omega(|p^\gamma y - n|_p) - \Omega(p|p^\gamma y - n - j|_p)) \\ &= \Omega(|p^\gamma x - n|_p) (\Omega(|p^\gamma y - n|_p) - \Omega(p|p^\gamma y - p^\gamma x|_p)) = \Omega(|p^\gamma x - n|_p) \\ &\quad \times (\Omega(|p^\gamma y - p^\gamma x|_p) - \Omega(p|p^\gamma y - p^\gamma x|_p)) = \Omega(|p^\gamma x - n|_p) \delta_{1, |p^\gamma x - p^\gamma y|_p} \end{aligned}$$

□

Note that the operators T^{I} and T^{II} are identically equal for $p = 2$. Actually, in this case the expression (13) contains only $T^{(\gamma n 0 1)}$ and $T^{(\gamma n 1 0)}$. Since $T^{(\gamma n 0 1)} = T^{(\gamma n 1 0)}$, from the proposition 4 it follows that $T_{xy}^{\text{I}} = T_{xy}^{\text{II}}$.

Now we will construct the basis of eigenfunctions of the operator T^{II} .

Consider the $p \times p$ matrix $\left(\mathbf{W}^{(\gamma n)}\right)_{ls}$ with matrix elements equal to $-T^{(\gamma n j k)}$ for $j \neq k$ and equal to $\sum_{\substack{k=0 \\ k \neq l}}^{p-1} T^{(\gamma n k l)}$ for the diagonal elements:

$$\mathbf{W}_{ls}^{(\gamma n)} = \delta_{sl} \sum_{\substack{k=0 \\ k \neq l}}^{p-1} T^{(\gamma n l k)} - (1 - \delta_{sl}) T^{(\gamma n l s)} \quad (18)$$

It is easy to see that $\mathbf{W}^{(\gamma n)}$ is a real symmetric $p \times p$ matrix. Moreover, the matrix is positive:

Lemma 5 The matrix $\mathbf{W}^{(\gamma n)}$ defined by (18) is positive.

Proof Compute the Hermitian form (we omit the (γn) index)

$$\langle z, \mathbf{W}z \rangle = \sum_{s=0}^{p-1} |z_s|^2 \sum_{\substack{l=0 \\ l \neq s}}^{p-1} W_{sl} - \sum_{s=0}^{p-1} \sum_{\substack{l=0 \\ l \neq s}}^{p-1} z_s^* z_l W_{sl} = \sum_{s=0}^{p-1} \sum_{\substack{l=0 \\ l \neq s}}^{p-1} W_{sl} (|z_s|^2 - z_s^* z_l)$$

Combining the terms containing W_{sl} and W_{ls} and using that $T^{(ls)}$ is a real symmetric matrix with nonnegative entries, we obtain for the combination of the terms

$$T_{sl}(|z_s|^2 + |z_l|^2 - z_s^* z_l - z_l^* z_s) \geq 0$$

that proves the positivity and finishes the proof of the lemma. \square

The matrix $\mathbf{W}^{(\gamma n)}$ has p nonnegative eigenvalues. Let us assume that the j -th ($j = 1, \dots, p-1$) eigenvector has the coordinates $h_{\gamma nj}^k$, $k = 0, \dots, p-1$ and corresponds to the eigenvalue $\lambda_j^{(\gamma n)}$:

$$\sum_{k=0}^{p-1} W_{lk}^{(\gamma n)} h_{\gamma nj}^k = \lambda_j^{(\gamma n)} h_{\gamma nj}^l$$

It is easy to see that the matrix $\mathbf{W}^{(\gamma n)}$ has the zero eigenvalue, which corresponds to the eigenvector with equal matrix elements: $h_{\gamma n0}^k = p^{-1/2}$ for all $k = 0, \dots, p-1$. Since eigenvectors are orthonormal we have

$$\sum_{k=0}^{p-1} h_{\gamma nj}^{*k} h_{\gamma nj'}^k = \delta_{jj'}, \quad j, j' = 0, \dots, p-1$$

Let us represent all eigenvectors of the matrix $\mathbf{W}^{(\gamma n)}$ in the form of the discrete Fourier expansion:

$$h_{\gamma nj}^k = \sum_{s=0}^{p-1} \chi_p(p^{-1}k(n+s)) g_{\gamma nj}^s, \quad j = 0, \dots, p-1 \quad (19)$$

From the properties of the discrete Fourier transformation and the properties of eigenvectors of the matrix $\mathbf{W}^{(\gamma n)}$, it follows that

$$g_{\gamma nj}^s = \sum_{k=0}^{p-1} h_{\gamma nj}^k \chi_p(-p^{-1}k(n+s)) \quad (20)$$

$$g_{\gamma nj}^0 = g_{\gamma n0}^j = 0, \quad j = 0, \dots, p-1 \quad (21)$$

$$\sum_{s=1}^{p-1} g_{\gamma nj}^s g_{\gamma nj'}^{s*} = \sum_{k=0}^{p-1} h_{\gamma nj}^k h_{\gamma nj'}^{k*} = \delta_{jj'} \quad (22)$$

Consider the function $\psi_{\gamma nj}(x)$ of the form

$$\psi_{\gamma nj}(x) = p^{\frac{1-\gamma}{2}} \sum_{k=0}^{p-1} h_{\gamma nj}^k \Omega(p|p^\gamma x - n - k|_p) \quad (23)$$

$$\gamma \in \mathbb{Z}, \quad n \in \mathbb{Q}_p/\mathbb{Z}_p, \quad j = 1, \dots, p-1$$

Note that $\psi_{\gamma nj}(x)$ is a locally constant function, constant on disks of the radius $p^{\gamma-1}$ and $\psi_{\gamma nj}(x) \in L^2(\mathbb{Q}_p)$. The functions $\psi_{\gamma nj}(x)$ we will call the generalized p -adic wavelets.

Using the expansion (19), the relations (20), (21), (22) and the formula (17), for $\psi_{\gamma nj}(x)$, we have

$$\begin{aligned}\psi_{\gamma nj}(x) &= p^{-\gamma/2} \sum_{k=0}^{p-1} \sum_{s=1}^{p-1} \chi_p(p^{-1}s(n+k)) g_{\gamma nj}^s \Omega(p|p^\gamma x - n - k|_p) \\ &= p^{-\gamma/2} \sum_{s=1}^{p-1} \chi_p(p^{\gamma-1}sx) g_{\gamma nj}^s \sum_{k=0}^{p-1} \Omega(p|p^\gamma x - n - k|_p) \\ &= \sum_{s=1}^{p-1} g_{\gamma nj}^s p^{-\gamma/2} \chi_p(p^{\gamma-1}sx) \Omega(p|p^\gamma x - n|_p) = \sum_{s=1}^{p-1} g_{\gamma nj}^s \varphi_{\gamma ns}(x)\end{aligned}$$

Thus, each of the functions (19) can be represented in the form of linear combination of p -adic wavelets (7):

$$\psi_{\gamma nj}(x) = \sum_{s=1}^{p-1} g_{\gamma nj}^s \varphi_{\gamma ns}(x) \quad (24)$$

For further computations we use the following identity

$$\begin{aligned}\Omega(p|p^\gamma x - n - k|_p) \Omega(p|p^{\gamma'} x - n' - l|_p) \\ = \delta_{p^{\gamma'-\gamma}(n+k), n'+l} \theta(\gamma' - \gamma) \Omega(p|p^\gamma x - n - k|_p) \\ + \delta_{n+k, p^{\gamma-\gamma'}(n'+l)} (1 - \theta(\gamma' - \gamma)) \Omega(p|p^{\gamma'} x - n' - l|_p)\end{aligned} \quad (25)$$

where

$$\theta(\gamma) = \begin{cases} 1, & \gamma > 0 \\ 0, & \gamma \leq 0 \end{cases}$$

and $\delta_{p^{\gamma'-\gamma}(n+k), n'+l}$ is the Kronecker symbol on the group $\mathbb{Q}_p/p\mathbb{Z}_p$:

$$\Omega(p|p^{\gamma'-\gamma}(n+k) - (n'+l)|_p) = \delta_{p^{\gamma'-\gamma}(n+k), n'+l} \quad \text{for } \gamma' \geq \gamma$$

Theorem 6 The set of functions $\{\psi_{\gamma nj}(x)\}$, $\gamma \in \mathbb{Z}$, $n \in \mathbb{Q}_p/\mathbb{Z}_p$, $j = 1, \dots, p-1$ is an orthonormal basis in $L^2(\mathbb{Q}_p)$.

Proof Consider the scalar product

$$\begin{aligned}p^{\frac{\gamma-1}{2}} p^{\frac{\gamma'-1}{2}} \langle \psi_{\gamma nj}, \psi_{\gamma' n' j'} \rangle \\ = \int_{\mathbb{Q}_p} \sum_{s=0}^{p-1} h_{\gamma nj}^{*s} \Omega(p|p^\gamma x - n - s|_p) \sum_{l=0}^{p-1} h_{\gamma' n' j'}^l \Omega(p|p^{\gamma'} x - n' - l|_p) d\mu(x)\end{aligned}$$

Using (25) we compute the following

$$\begin{aligned}p^{\frac{\gamma-1}{2}} p^{\frac{\gamma'-1}{2}} \langle \psi_{\gamma nj}, \psi_{\gamma' n' j'} \rangle &= \sum_{s=0}^{p-1} h_{\gamma nj}^{*s} \sum_{l=0}^{p-1} h_{\gamma' n' j'}^l \int_{\mathbb{Q}_p} \left(\delta_{p^{\gamma'-\gamma}(n+s), n'+l} \theta(\gamma' - \gamma) \Omega(p|p^\gamma x - n - s|_p) \right. \\ &\quad \left. + \delta_{n+s, p^{\gamma-\gamma'}(n'+l)} (1 - \theta(\gamma' - \gamma)) \Omega(p|p^{\gamma'} x - n' - l|_p) \right) d\mu(x) \\ &= \sum_{s=0}^{p-1} h_{\gamma nj}^{*s} \sum_{l=0}^{p-1} h_{\gamma' n' j'}^l \left(\delta_{p^{\gamma'-\gamma}(n+s), n'+l} \theta(\gamma' - \gamma) p^{\gamma-1} + \delta_{n+s, p^{\gamma-\gamma'}(n'+l)} (1 - \theta(\gamma' - \gamma)) p^{\gamma'-1} \right)\end{aligned}$$

In this expression, if the first term is non zero and $\gamma' > \gamma$, then $\delta_{p^{\gamma'-\gamma}(n+s), n'+l}$ does not depend on s and we obtain the summation on s of the form $\sum_{s=0}^{p-1} h_{\gamma nj}^{*s}$ which is equal to zero for $j > 0$.

In the same way, we prove that the second term does not vanish only for $\gamma' = \gamma$. This proves that

$$p^{\frac{\gamma-1}{2}} p^{\frac{\gamma'-1}{2}} \langle \psi_{\gamma nj}, \psi_{\gamma' n' j'} \rangle = \delta_{\gamma \gamma'} \delta_{nn'} p^{\gamma-1} \sum_{s=0}^{p-1} h_{\gamma nj}^{*s} h_{\gamma nj'}^s = \delta_{\gamma \gamma'} \delta_{nn'} \delta_{jj'} p^{\gamma-1} \|h_{(\gamma n)j}\|^2$$

which implies that $\{\psi_{\gamma nj}\}$ is an orthonormal system of functions.

To prove that the set of vectors $\{\psi_{\gamma nj}\}$ is an orthonormal basis (is total in $L^2(\mathbb{Q}_p)$) we use the Parsevale identity. Since the set of indicators (characteristic functions) of p -adic disks is total in $L^2(\mathbb{Q}_p)$ it is enough to check the Parsevale identity for the indicator $\Omega(p|p^\gamma x - n - s|_p)$. We have for the scalar product of the indicator and the wavelet

$$\langle \Omega(p|p^\gamma x - n - s|_p), \psi_{\gamma' n' j'} \rangle = p^{\frac{1-\gamma'}{2}} \sum_{l=0}^{p-1} h_{\gamma' n' j'}^l p^{\gamma-1} \left(\delta_{p^{\gamma'-\gamma} n, n'+l} \theta(\gamma' - \gamma) + \delta_{\gamma \gamma'} \delta_{nn'} \delta_{sl} \right)$$

Summing up the wavelets we get

$$\begin{aligned} & \sum_{\gamma' n' j'} |\langle \Omega(p|p^\gamma x - n - s|_p), \psi_{\gamma' n' j'}(x) \rangle|^2 \\ &= p^{\gamma-1} \left[\sum_{j'} |h_{\gamma nj'}^s|^2 + p^{\gamma-1} \sum_{\gamma' > \gamma; n' j'} p^{1-\gamma'} \sum_{l=0}^{p-1} |h_{\gamma' n' j'}^l|^2 \delta_{p^{\gamma'-\gamma} n, n'+l} \right] \end{aligned} \quad (26)$$

Using the normalization condition we get

$$\sum_{j=1}^{p-1} h_{\gamma nj}^{*s} h_{\gamma nj}^s = 1 - p^{-1} \quad (27)$$

which implies for (26)

$$(1 - p^{-1}) p^{\gamma-1} \left[1 + p^{\gamma-1} \sum_{\gamma' > \gamma} p^{1-\gamma'} \right] = (1 - p^{-1}) p^{\gamma-1} (1 - p^{-1})^{-1} = p^{\gamma-1}$$

that proves the Parsevale identity. \square

In the next theorem we prove that the constructed in the theorem above basis is an eigenbasis of the ultrametric diffusion operator T^{II} and compute the corresponding eigenvalues.

Theorem 7 *Let the kernel (13) satisfies the condition of convergence of the series*

$$\sum_{\gamma > 0} \sum_{k=1}^{p-1} p^\gamma T^{(\gamma 0 k 0)} \quad (28)$$

Then the operator (2) is a well defined operator in $L^2(\mathbb{Q}_p)$ with the dense domain and the generalized p -adic wavelets $\psi_{\gamma nj}$ are eigenvectors for the operator T^Π :

$$T^\Pi \psi_{\gamma nj} = \lambda_{\gamma nj}^\Pi \psi_{\gamma nj}$$

with the eigenvalues

$$\lambda_{\gamma nj}^\Pi = p^{\gamma-1} \lambda_j^{(\gamma n)} + \sum_{\substack{\gamma' n' lk \\ l \neq k}} p^{\gamma'-1} T^{(\gamma' n' lk)} \delta_{p^{\gamma'-\gamma n, n'+l}} \theta(\gamma' - \gamma) \quad (29)$$

Proof Consider the action of the operator on the wavelet $\psi_{\gamma nj}$.

$$\begin{aligned} p^{\frac{\gamma-1}{2}} T^\Pi \psi_{\gamma nj}(x) &= p^{\frac{\gamma-1}{2}} \int T_{xy}^\Pi (\psi_{\gamma nj}(x) - \psi_{\gamma nj}(y)) d\mu(y) \\ &= p^{\gamma'-1} \sum_{\substack{\gamma' n' lk \\ l \neq k}} T^{(\gamma' n' lk)} \Omega(p|p^{\gamma'} x - n' - l|_p) \sum_{s=0}^{p-1} h_{\gamma nj}^s \Omega(p|p^\gamma x - n - s|_p) \\ &\quad - \sum_{\substack{\gamma' n' lk \\ l \neq k}} T^{(\gamma' n' lk)} \Omega(p|p^{\gamma'} x - n' - l|_p) \sum_{s=0}^{p-1} h_{\gamma nj}^s \\ &\quad \times \int \Omega(p|p^{\gamma'} y - n' - k|_p) \Omega(p|p^\gamma y - n - s|_p) d\mu(y) \end{aligned}$$

Using (25) for $p^{\frac{\gamma-1}{2}} T^\Pi \psi_{\gamma nj}(x)$, we compute the following

$$\begin{aligned} &\sum_{\substack{\gamma' n' lk \\ l \neq k}} T^{(\gamma' n' lk)} \sum_{s=0}^{p-1} h_{\gamma nj}^s \left[\delta_{p^{\gamma'-\gamma(n+s), n'+l}} \theta(\gamma' - \gamma) p^{\gamma'-1} \Omega(p|p^\gamma x - n - s|_p) \right. \\ &\quad + \left[\delta_{n+s, p^{\gamma-\gamma'}(n'+l)} - \delta_{n+s, p^{\gamma-\gamma'}(n'+k)} \right] (1 - \theta(\gamma' - \gamma)) p^{\gamma'-1} \Omega(p|p^{\gamma'} x - n' - l|_p) \\ &\quad \left. - \delta_{p^{\gamma'-\gamma(n+s), n'+k}} \theta(\gamma' - \gamma) p^{\gamma'-1} \Omega(p|p^{\gamma'} x - n' - l|_p) \right] \end{aligned}$$

We prove that the term proportional to $1 - \theta(\gamma' - \gamma)$ is equal to the following

$$\left[\delta_{n+s, p^{\gamma-\gamma'}(n'+l)} - \delta_{n+s, p^{\gamma-\gamma'}(n'+k)} \right] (1 - \theta(\gamma' - \gamma)) = \delta_{\gamma\gamma'} \delta_{nn'} (\delta_{sl} - \delta_{sk})$$

and

$$\sum_{s=0}^{p-1} h_{\gamma nj}^s \delta_{p^{\gamma'-\gamma(n+s), n'+k}} \theta(\gamma' - \gamma) = \delta_{p^{\gamma'-\gamma n, n'+k}} \theta(\gamma' - \gamma) \sum_{s=0}^{p-1} h_{\gamma nj}^s = 0$$

since for $j = 1, \dots, p-1$, we have $\sum_{s=0}^{p-1} h_{\gamma nj}^s = 0$.

This implies for $p^{\frac{\gamma-1}{2}} T^\Pi \psi_{\gamma nj}(x)$ the following

$$\begin{aligned} p^{\frac{\gamma-1}{2}} T^\Pi \psi_{\gamma nj}(x) &= \left[\sum_{\substack{\gamma' n' lk \\ l \neq k}} T^{(\gamma' n' lk)} \delta_{p^{\gamma'-\gamma n, n'+l}} \theta(\gamma' - \gamma) p^{\gamma'-1} \right] \sum_{s=0}^{p-1} h_{\gamma nj}^s \Omega(p|p^\gamma x - n - s|_p) \\ &\quad + p^{\gamma-1} \sum_{s=0}^{p-1} h_{\gamma nj}^s \sum_{\substack{lk \\ l \neq k}} T^{(\gamma n lk)} (\delta_{sl} - \delta_{sk}) \Omega(p|p^\gamma x - n - l|_p) \end{aligned}$$

Consider

$$\begin{aligned}
& \sum_{s=0}^{p-1} h_{\gamma nj}^s \sum_{\substack{lk \\ l \neq k}} T^{(\gamma nlk)} (\delta_{sl} - \delta_{sk}) \Omega(p|p^\gamma x - n - l|_p) \\
&= \sum_{l=0}^{p-1} \Omega(p|p^\gamma x - n - l|_p) \sum_{s=0}^{p-1} \left(\delta_{sl} \sum_{\substack{lk \\ l \neq k}} T^{(\gamma nlk)} - (1 - \delta_{sl}) T^{(\gamma nls)} \right) h_{\gamma nj}^s \\
&= \sum_{l=0}^{p-1} \Omega(p|p^\gamma x - n - l|_p) \lambda_j^{(\gamma n)} h_{\gamma nj}^l = p^{\frac{\gamma-1}{2}} \lambda_j^{(\gamma n)} \psi_{\gamma nj}(x)
\end{aligned}$$

Finally, we obtain

$$T^\Pi \psi_{\gamma nj}(x) = \left[p^{\gamma-1} \lambda_j^{(\gamma n)} + \sum_{\substack{\gamma' n' lk \\ l \neq k}} p^{\gamma'-1} T^{(\gamma' n' lk)} \delta_{p^{\gamma'-\gamma} n, n'+l} \theta(\gamma' - \gamma) \right] \psi_{\gamma nj}(x)$$

Using the condition of convergence of the series (28) we obtain the proof of the theorem. \square

4. Relaxation problem

Let us consider the relaxation problem formulated analogously to that in [6–8] (see also references therein). In these works, the evolution of probability distribution was described by the equation of the form (1) with the ultrametric diffusion operator of the type T^0 . The initial distribution was taken homogeneous on \mathbb{Z}_p (i.e. $f(x, 0) = \Omega(|x|_p)$).

Consider the relaxation function of the system, $R(t)$, which describes the evolution of population of the system in the set where the initial distribution was concentrated. In the case when the initial distribution is the characteristic function of the unit ball with the center in zero, this reduces to

$$R(t) = \langle \Omega(|x|_p), e^{-Tt} \Omega(|x|_p) \rangle = \int_{\mathbb{Z}_p} f(x, t) d\mu(x)$$

It is known (see, in particular, [1, 6–8, 21]), for the case when the ultrametric diffusion is generated by T^0 , the relaxation function $R(t)$ takes the form

$$R^0(t) = \left\langle \Omega(|x|_p), e^{-T^0 t} \Omega(|x|_p) \right\rangle = (p-1) \sum_{\gamma=1}^{\infty} p^{-\gamma} \exp(-\lambda_\gamma^0 t) \quad (30)$$

Let us investigate the relaxation behavior for the cases, when the ultrametric diffusion is generated by the operators T^I and T^Π . The initial distribution we will take to be equal to the characteristic function of the disk:

$$f(x, 0) = \Omega(|x - a|_p), \quad |a|_p = p^N, \quad N \geq 1 \quad (31)$$

and consider the relaxation function

$$R^I(t) = \left\langle \Omega(|x - a|_p), e^{-T^I t} \Omega(|x - a|_p) \right\rangle$$

and R^{II} , defined analogously.

The operator T^I

Find the coefficients of the decomposition of the initial condition (31) in the basis of p -adic wavelets (7).

$$C_{\gamma nj} = \langle \Omega(|x - a|_p), \varphi_{\gamma nj}(x) \rangle = \varphi_{\gamma nj}(a) \theta(\gamma)$$

The solution of the Cauchy problem for the ultrametric diffusion equation (1) with the operator T^I and the initial condition (31) takes the form

$$f^I(x, t) = \sum_{\gamma=1}^{\infty} \sum_{n \in \mathbb{Q}_p/\mathbb{Z}_p} \sum_{j=1}^{p-1} e^{-\lambda_{\gamma n}^I t} \varphi_{\gamma nj}(a) \varphi_{\gamma nj}^*(x)$$

Then the relaxation function $R^I(t)$ is given by the expression

$$\begin{aligned} R^I(t) &= \sum_{\gamma=1}^{\infty} \sum_{n \in \mathbb{Q}_p/\mathbb{Z}_p} \sum_{j=1}^{p-1} e^{-\lambda_{\gamma n}^I t} |\varphi_{\gamma nj}(a)|^2 \\ &= (p-1) \sum_{\gamma=1}^{\infty} \sum_{n \in \mathbb{Q}_p/\mathbb{Z}_p} p^{-\gamma} e^{-\lambda_{\gamma n}^I t} \Omega(|p^\gamma a - n|_p) \end{aligned}$$

Taking into account that $|a|_p = p^N$ and $N \geq 1$, let us divide the series into the two parts: for $\gamma \leq N$ and for $\gamma > N$. For the second part the values of all the indicators are equal to one only if $n = 0$, otherwise they are equal to zero. Hence, for the relaxation function $R^I(t)$ we get

$$R^I(t) = (p-1) \sum_{\gamma=1}^N p^{-\gamma} e^{-t\lambda_{\gamma, p^\gamma a}^I} + (p-1) \sum_{\gamma=N+1}^{\infty} p^{-\gamma} e^{-t\lambda_{\gamma 0}^I} \quad (32)$$

The operator T^{II}

The solution of the Cauchy problem for the ultrametric diffusion equation (1) with the operator T^{II} and the initial condition (31) takes the form

$$f^{II}(x, t) = \sum_{\gamma=1}^{\infty} \sum_{n \in \mathbb{Q}_p/\mathbb{Z}_p} \sum_{j=1}^{p-1} e^{-\lambda_{\gamma nj}^{II} t} \psi_{\gamma nj}^*(x) \psi_{\gamma nj}(a)$$

Then the relaxation function $R^{II}(t)$ is given by the expression

$$\begin{aligned} R^{II}(t) &= \sum_{\gamma=1}^{\infty} \sum_{n \in \mathbb{Q}_p/\mathbb{Z}_p} \sum_{j=1}^{p-1} e^{-\lambda_{\gamma nj}^{II} t} |\psi_{\gamma nj}(a)|^2 \\ &= \sum_{\gamma=1}^{\infty} \sum_{n \in \mathbb{Q}_p/\mathbb{Z}_p} \sum_{j=1}^{p-1} e^{-\lambda_{\gamma nj}^{II} t} p^{1-\gamma} \sum_{k=0}^{p-1} |h_{\gamma nj}^k|^2 \Omega(p|p^\gamma a - n - k|_p) \end{aligned}$$

Taking into account that $|a|_p = p^N$ and $N \geq 1$, for the relaxation function $R^{\text{II}}(t)$ we get

$$R^{\text{II}}(t) = \sum_{\gamma=1}^N \sum_{j=1}^{p-1} \sum_{k=0}^{p-1} e^{-\lambda_{\gamma n j}^{\text{II}} t} p^{1-\gamma} |h_{\gamma n j}^k|^2 \delta_{p^{\gamma} a - k, n} + \sum_{\gamma=N+1}^{\infty} p^{-\gamma} \sum_{j=1}^{p-1} p |h_{\gamma 0 j}^0|^2 e^{-\lambda_{\gamma 0 j}^{\text{II}} t}$$

Comparing the formulas (30) and (32), we see that the long-time relaxation behavior for ultrametric diffusions generated by the operators T^0 and T^I coincide (i.e. the functions $R^0(t)$ and $R^I(t)$ have the same asymptotic). This shows that the particular properties of the energy landscape, such as local inhomogeneities, are not important for long time behavior of the corresponding diffusion. Note that the given result generalizes the special case considered in the work [22] by Yoshino.

For the cases of the operators of the type T^{II} , the relaxation function behavior becomes more complicated. Note that if the landscape deviations from the regularity are small:

$$|\lambda_{\gamma 0 j}^{\text{II}} - \langle \lambda_{\gamma 0 j}^{\text{II}} \rangle_j| \ll \langle \lambda_{\gamma 0 j}^{\text{II}} \rangle_j$$

where $\langle \lambda_{\gamma 0 j}^{\text{II}} \rangle_j = (p-1)^{-1} \sum_{j=1}^{p-1} \lambda_{\gamma 0 j}^{\text{II}}$, then by (27) the long-time relaxation $R^I(t)$ is a good approximation of the relaxation $R^{\text{II}}(t)$.

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